

同步练习 P206 1

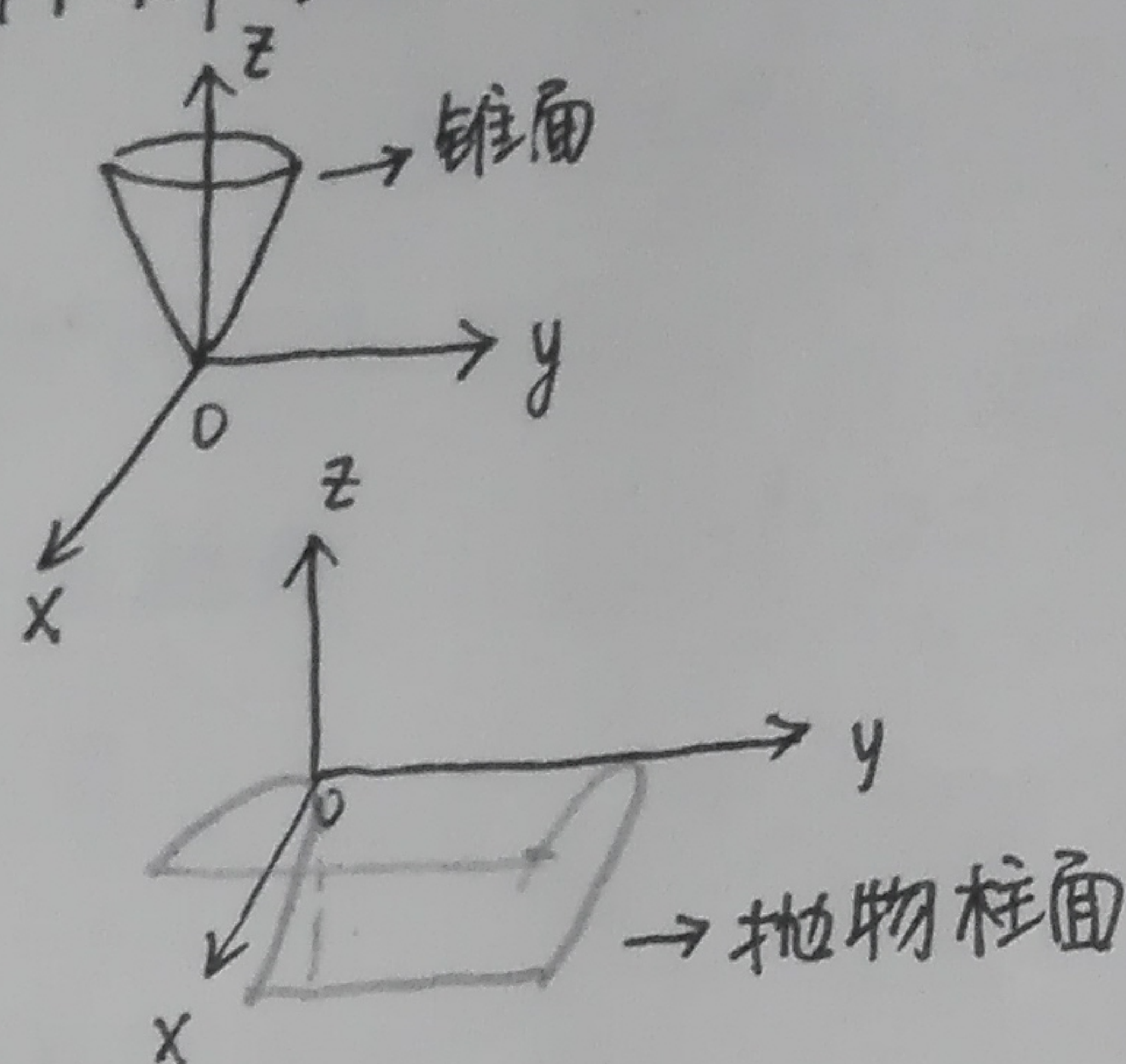
求锥面 $z = \sqrt{x^2 + y^2}$ 被柱面 $z^2 = 2x$ 所割下部分的曲面面积

解: $\begin{cases} z = \sqrt{x^2 + y^2} \\ z^2 = 2x \end{cases}$ 消去 z 得 $(x-1)^2 + y^2 = 1$

于是, 所割下部分的曲面在 xoy 平面上的
投影区域 D_{xy} : $(x-1)^2 + y^2 \leq 1$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{2}$$

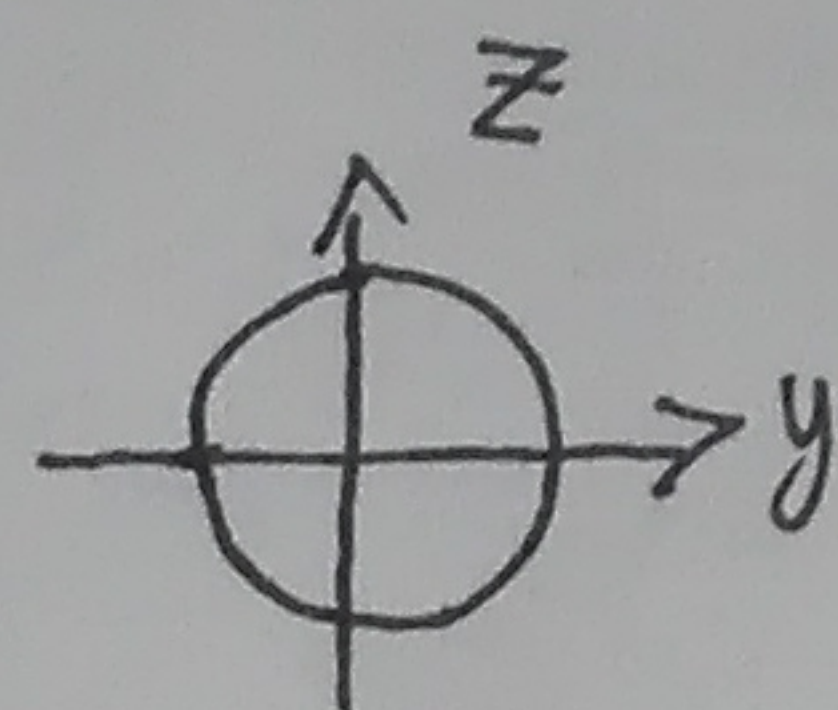
$$S = \iint_{D_x} \sqrt{2} \, dx \, dy = \sqrt{2} S_{D_x} = \sqrt{2} \pi$$



同步练习 Prob 2.

求曲面 $x = yz$ 被柱面 $y^2 + z^2 = 1$ 所割下部分的曲面面积

解: 割下部分曲面在 yz 平面上的投影



区域 $D_{yz}: y^2 + z^2 \leq 1$

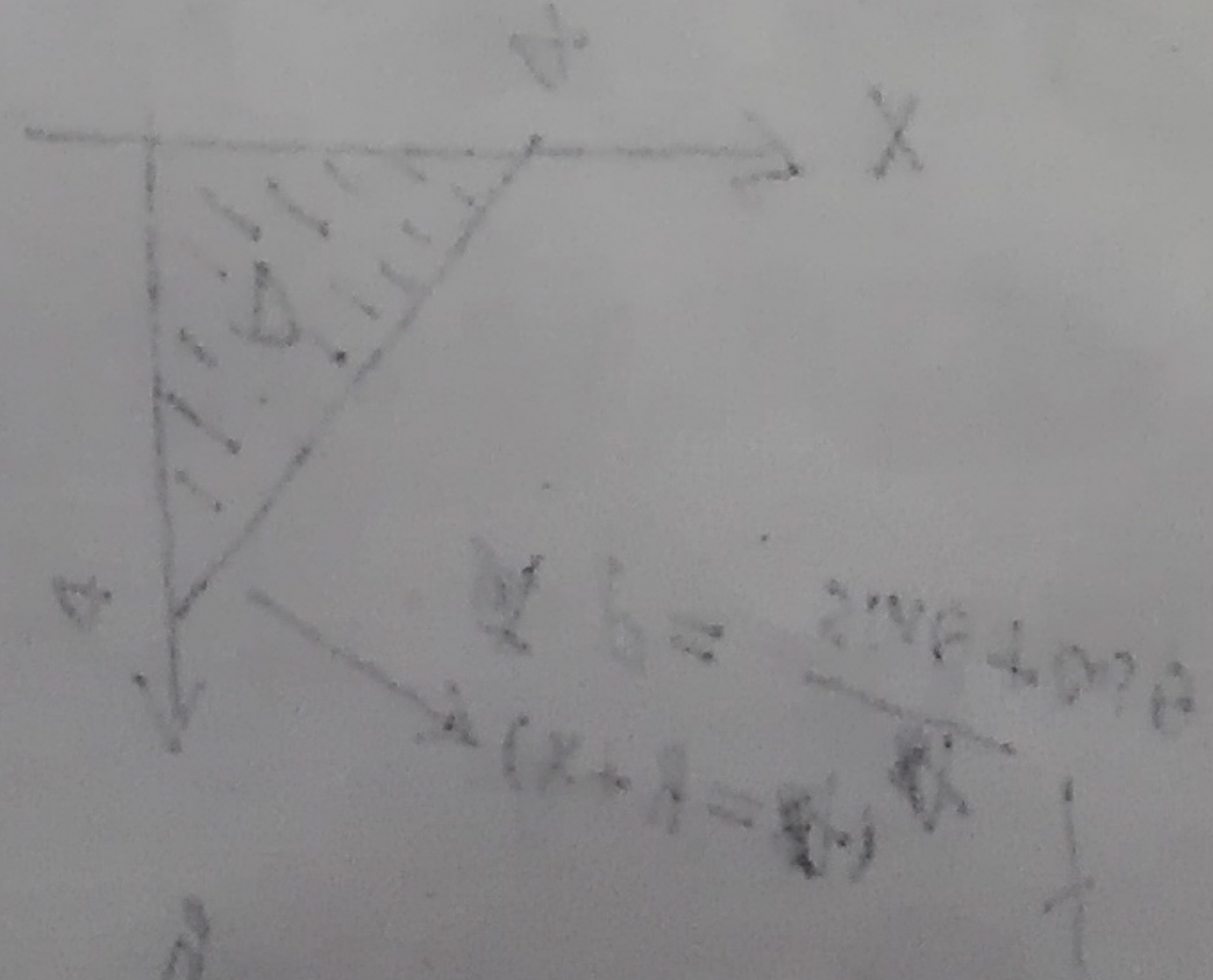
$$\sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2} = \sqrt{1 + z^2 + y^2}$$

$$S = \iint_{D_{yz}} \sqrt{1 + y^2 + z^2} \, dy \, dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 \sqrt{1 + \rho^2} \, \rho \, d\rho$$

$$= 2\pi \int_0^1 (1 + \rho^2)^{\frac{1}{2}} \cdot \frac{1}{2} \cdot d(1 + \rho^2)$$

$$= \pi \cdot \frac{2}{3} (1 + \rho^2)^{\frac{3}{2}} \Big|_0^1 = \frac{2\pi}{3} (2\sqrt{2} - 1)$$



同步练习 P206.3

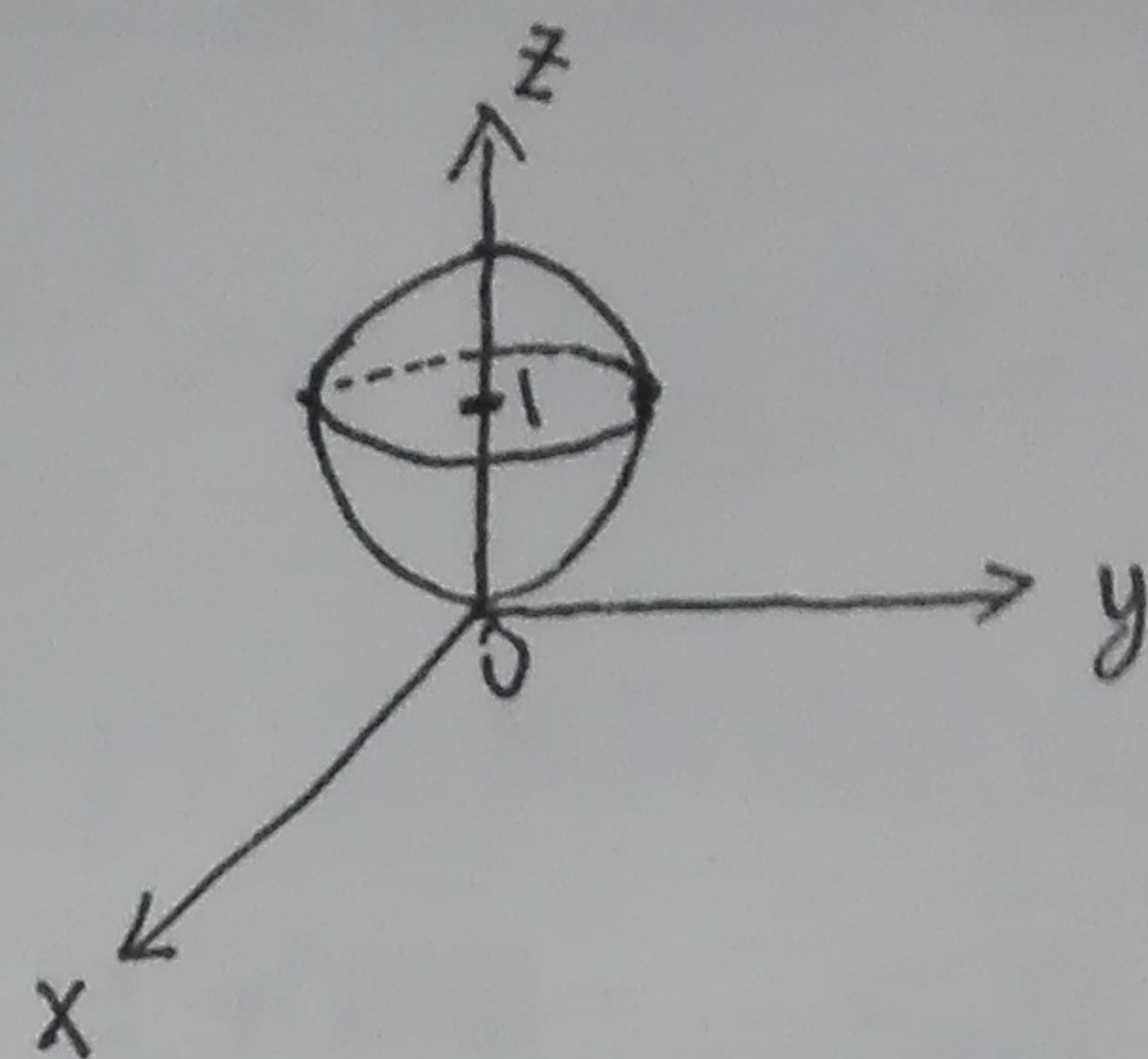
设球体 $x^2+y^2+z^2 \leq 2z$ 上各点的密度等于该点到坐标原点的距离平方

求该球体的质量

解: $\rho(x,y,z) = x^2+y^2+z^2$

球体区域 Ω 用球面坐标表示

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \frac{\pi}{2} \quad 0 \leq r \leq 2\cos\varphi$$



于是, $m = \iiint_{\Omega} \rho(x,y,z) dx dy dz$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2 \cdot r \sin\varphi dr$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \frac{32}{5} \cos^5\varphi \sin\varphi d\varphi$$

~~$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \frac{32}{5} \cos^5\varphi \sin\varphi d\varphi$$~~

~~$$= \frac{64\pi}{5} \int_0^{\frac{\pi}{2}} \cos^5\varphi \sin\varphi d\varphi$$~~

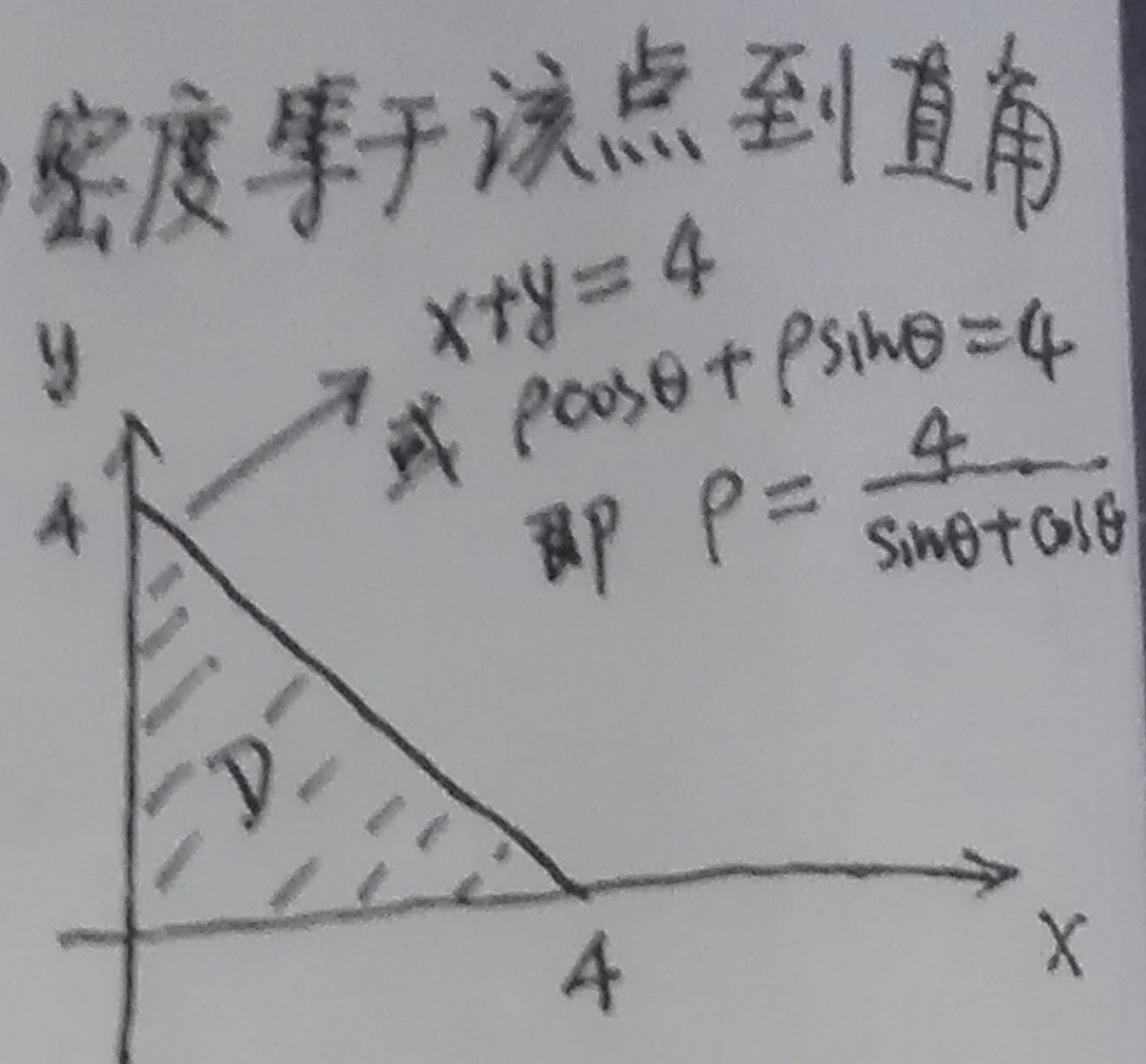
$$= \frac{64\pi}{5} \int_0^{\frac{\pi}{2}} \cos^5\varphi \sin\varphi d\varphi$$

$$= \frac{64}{5} \pi \cdot \frac{1}{6} = \frac{32}{15} \pi$$

同步练习 P207 5

设有一等腰直角三角形薄片, 腰长为4, 各点处的面密度等于该点到直角顶点距离平方, 求该薄片的质心。

解: 以两直角边为轴建立如图的直角坐标系。



$$D: \begin{aligned} 0 &\leq \theta \leq \frac{\pi}{2} \\ 0 &\leq \rho \leq \frac{4}{\sin\theta + \cos\theta} \end{aligned}$$

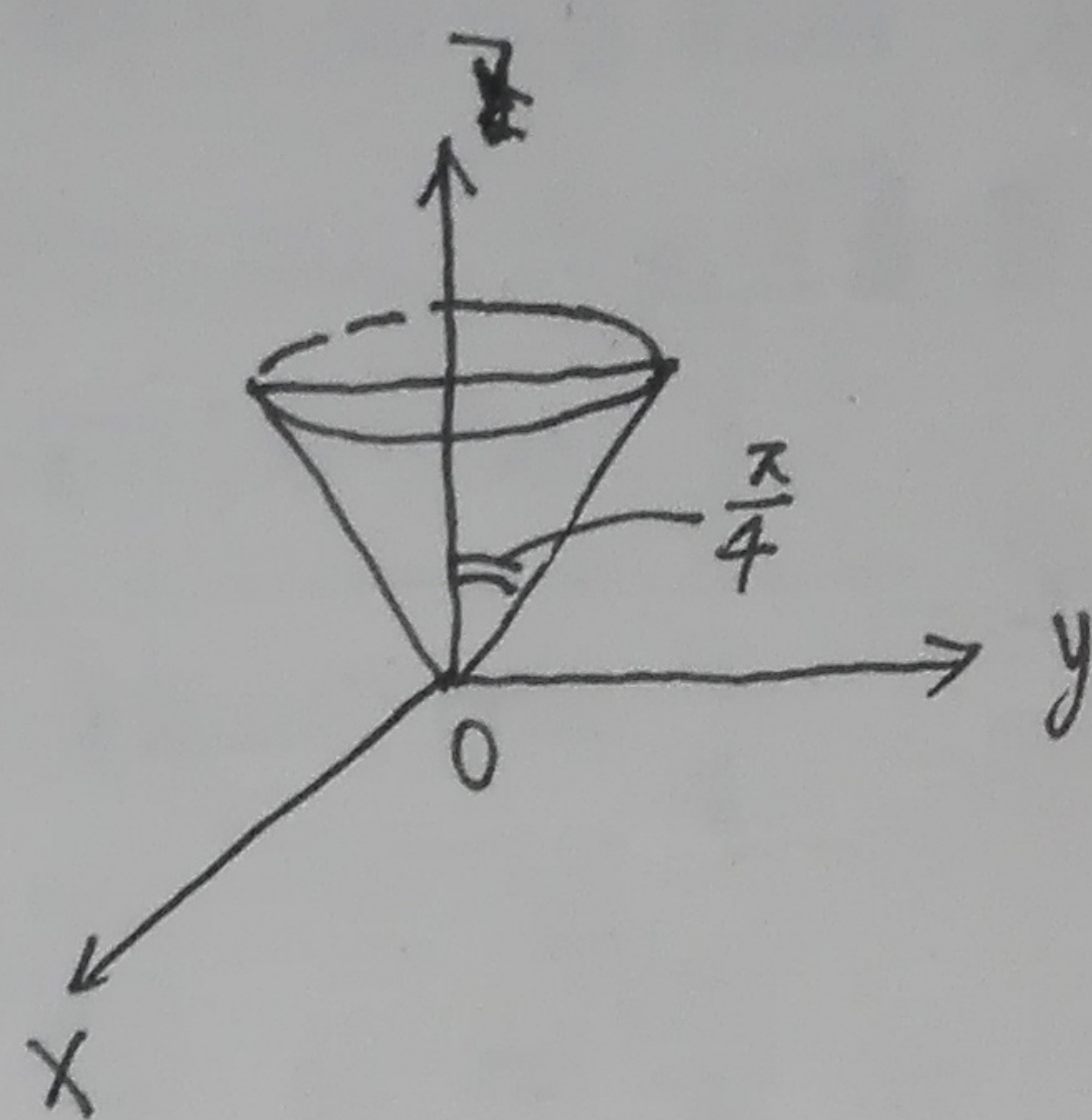
面密度函数 $u(x,y) = x^2 + y^2$, 由对称性知

$$\begin{aligned} \bar{x} &= \frac{\iint_D x(x^2+y^2) dx dy}{\iint_D (x^2+y^2) dx dy} = \frac{\iint_D y(x^2+y^2) dx dy}{\iint_D (x^2+y^2) dx dy} = \bar{y} \\ &= \frac{\bar{x} + \bar{y}}{2} = \frac{1}{2} \frac{\iint_D (x+y)(x^2+y^2) dx dy}{\iint_D (x^2+y^2) dx dy} \\ &= \frac{1}{2} \frac{\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{4}{\sin\theta + \cos\theta}} (\sin\theta + \cos\theta) \rho \cdot \rho^2 \rho d\rho}{\int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{4}{\sin\theta + \cos\theta}} \rho^2 \rho d\rho} \\ &= \frac{1}{2} \frac{\int_0^{\frac{\pi}{2}} (\sin\theta + \cos\theta) d\theta \cdot \left. \frac{\rho^5}{5} \right|_0^{\frac{4}{\sin\theta + \cos\theta}}}{\int_0^{\frac{\pi}{2}} d\theta \cdot \left. \frac{\rho^4}{4} \right|_0^{\frac{4}{\sin\theta + \cos\theta}}} \\ &= \frac{1}{2} \cdot \frac{\frac{4^5}{5} \int_0^{\frac{\pi}{2}} \left(\frac{1}{\sin\theta + \cos\theta} \right)^4 d\theta}{\frac{4^4}{4} \int_0^{\frac{\pi}{2}} \left(\frac{1}{\sin\theta + \cos\theta} \right)^4 d\theta} = \frac{8}{5} \end{aligned}$$

于是, 质心坐标 $(\frac{8}{5}, \frac{8}{5})$

同步练习 P207 6

求高为 h , 半顶角为 $\frac{\pi}{4}$ 的均匀正圆锥体绕其对称轴旋转的转动惯量.
 解: 以对称轴为 z 轴, 圆锥底面所在平面^{平行} xoy 平面建立如图所示的直角坐标系. 设体密度为 u .



$$\Omega: 0 \leq z \leq h, \quad 0 \leq \rho \leq z, \quad 0 \leq \theta \leq 2\pi$$

$$I = \iiint_{\Omega} (x^2 + y^2) u \, dV$$

$$= \int_0^h dz \int_0^z \rho^2 \rho \, d\rho \int_0^{2\pi} u \, d\theta$$

$$= 2\pi u \int_0^h dz \int_0^z \rho^3 \, d\rho = \frac{\pi u}{2} \int_0^h z^4 \, dz$$

$$= \frac{\pi u}{2} \cdot \frac{z^5}{5} \Big|_0^h = \frac{\pi u h^5}{10}$$